MATH 2050C Lecture on 3/18/2020

Recall: Cauchy seq. (x_n)

Def$^{\underline{n}}$: $\forall \varepsilon > 0$, $\exists H = H(\varepsilon) \in \mathbb{N}$ s.t.

$$|x_n - x_m| < \varepsilon \qquad \forall n, m \geq H$$

$\lim (x_n) = x$

Def$^{\underline{n}}$: $\forall \varepsilon > 0$, $\exists K = K(\varepsilon) \in \mathbb{N}$ s.t.

$$|x_n - x| < \varepsilon \qquad \forall n \geq K$$

Remarks: (1) We do not need to know x in advance in the definition of Cauchy sequences.

(2) The "m" and "n" can be freely chosen (independent of each other) in the def$^{\underline{n}}$ of Cauchy seq.

E.g.) It's NOT enough to have, say, $|x_{n+1} - x_n| < \varepsilon \quad \forall n \geq H$.

[Ex: Find a "counterexample".]

★ **Cauchy Criteria:** (x_n) Cauchy $\Leftrightarrow$ (x_n) convergent ★

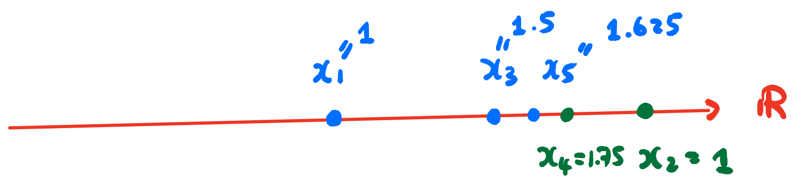
Note: Because it's an "if and only if" statement, one can use it to show (x_n) is convergent OR divergent.

Example: Let $x_1 := 1$, $x_2 := 2$, $x_n := \frac{1}{2}(x_{n-1} + x_{n-2}) \quad \forall n \geq 3$.

Show that (x_n) is convergent, and find $\lim (x_n)$.

Proof: Write out a few terms:

$$(x_n) = (1, 2, 1.5, 1.75, 1.625, \ldots\ldots)$$



M.I. $\Rightarrow$ $|x_{n+1} - x_n| = \frac{1}{2^{n-1}} \quad \forall n \in \mathbb{N}$

So, $|x_{n+2} - x_{n+1}| = \frac{1}{2}|x_{n+1} - x_n| \Rightarrow (x_n)$ contractive (w/. $C = \frac{1}{2}$)

Tutorial $\Rightarrow$ (x_n) Cauchy $\Rightarrow$ (x_n) convergent, say to some $x \in \mathbb{R}$.

To find x, the recursive relation $x_n = \frac{1}{2}(x_{n-1} + x_{n-2}) \Rightarrow \underbrace{x = \frac{1}{2}(x+x)}_{\text{NOT useful}}$.

Since $\lim(x_n) = x$, any subseq. (x_{n_k}) also converges to x.

Consider the odd subseq. $(x_{2k-1})_{k \in \mathbb{N}}$.

$$x_{2k-1} = 1 + \frac{1}{2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{2k-3}} = 1 + \frac{\frac{1}{2}\left(1 - \frac{1}{4^{k-1}}\right)}{1 - \frac{1}{4}}$$

$$\overset{\text{as } k\to\infty}{\Rightarrow} \quad x = \lim(x_{2k-1}) = 1 + \frac{1/2}{3/4} = 5/3 \quad *$$

Limit of functions (Ch. 4)

GOAL: Define $\lim_{x \to c} f(x)$ for functions $f: A \subseteq \mathbb{R} \to \mathbb{R}$

& study its properties.

Basic Idea: "$\lim_{x \to c} f(x) = L$" $\leftrightsquigarrow$ $f(x) \approx L$ whenever $x \approx c$

Need $x \in A$
to be defined

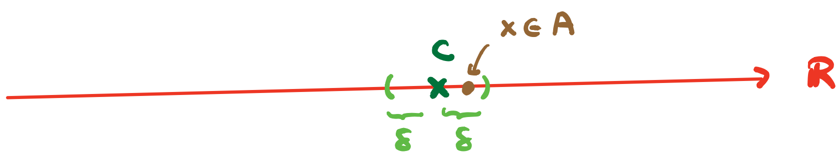
This motivates the following:

Def$^{\underline{n}}$: Let $A \subseteq \mathbb{R}$ be an arbitrary subset (not nec. closed / open).

We say $c \in \mathbb{R}$ is a cluster point of A if

$\forall \delta > 0$, $\exists\ x = x(\delta) \in A$ s.t. $|x - c| < \delta$ and $x \neq c$

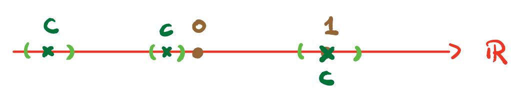
"small"



[Intuitively, $\exists$ points in A, other than c itself, close to c.]

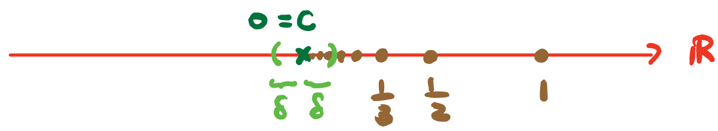
Examples:

- $A = \{0, 1\}$ has NO cluster pt.

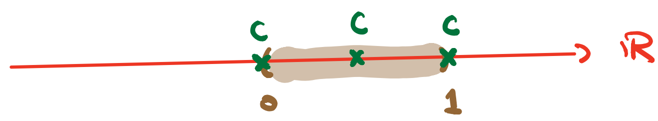


- $A = \{a_1, \ldots, a_N\}$ has NO cluster pt.
- $A = \mathbb{N}$ has NO cluster pt. (Ex: Prove this)

- $A = \{ \frac{1}{n} : n \in \mathbb{N} \}$ has 1 cluster pt. $c = 0$



- $A = (0,1)$ has cluster pts $c \in [0,1]$



<u>Remark</u>: A cluster pt $c \in \mathbb{R}$ may be $c \in A$ OR $c \notin A$.